

# $L^\infty$ -VARIATIONAL PROBLEMS ASSOCIATED TO MEASURABLE FINSLER STRUCTURES

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**ABSTRACT.** We study  $L^\infty$ -variational problems associated to measurable Finsler structures in Euclidean spaces. We obtain existence and uniqueness results for the absolute minimizers.

**Keywords:**  $L^\infty$ -variational problems; Existence; Uniqueness; Finsler structure

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## 1. INTRODUCTION

In this paper, we study the  $L^\infty$ -variational problem

$$\mathbb{F}(u; U) := \operatorname{ess\,sup}_{x \in U} F(x, \nabla u(x)) \quad (1.1)$$

over the class of Lipschitz functions on  $U \subset\subset \Omega$  with a given boundary data, where  $U \subset\subset \Omega$  is an arbitrary open subset of a given domain  $\Omega$  in the Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ , and  $F : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a Borel measurable Finsler structure on  $\Omega$  (see Definition 2.1 below). Above,  $\nabla u(x)$  denotes the gradient of  $u$  at  $x$ . By Rademacher's theorem, any

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locally Lipschitz continuous function is differentiable at almost every point, and hence (1.1) makes sense. For applications of this  $L^\infty$  calculus of variation, see [24, 22] and the references therein.

The study of  $L^\infty$ -variational problems of type (1.1) was initiated by Aronsson [2, 3, 4, 5] in the model case  $F(x, \xi) := |\xi|^2$ . That is, consider the functional (1.1) in the form

$$\mathbb{F}(u, U) := \operatorname{ess\,sup}_{x \in U} |\nabla u(x)|^2. \quad (1.2)$$

Since then the study of the  $L^\infty$ -variational problem, for more general functionals  $F$  with various smoothness assumptions, has advanced significantly; see the seminal works [10, 25] and the survey paper [6] for more information on the recent developments. The  $L^\infty$ -variational problem is also interesting even if the functional  $F$  is not smooth or even not continuous; see for example [6, 8, 9, 19, 30] and the references therein. In the following, we first briefly review some results on the model case (1.2). This model case is of great importance, due to its simple structure, and due to all the techniques that are developed to study the existence and the uniqueness of (1.2) can be possibly applied to the general functional of the form (1.1). Then we present the main result of this paper.

In the model case (1.2), Aronsson introduced the idea of absolute minimizers in his series of papers [2, 3, 4, 5]. His idea easily extends to the general case (1.1). Precisely, let  $U \subset \subset \Omega$  be an arbitrary open subset. Denote by  $\operatorname{Lip}(U)$  the space of Lipschitz continuous functions on  $U$  with respect to the standard Euclidean metric, and by  $\operatorname{Lip}_{\operatorname{loc}}(U)$  the space of locally Lipschitz continuous functions on  $U$ . A function  $u \in \operatorname{Lip}_{\operatorname{loc}}(U) \cap C(\overline{U})$  is called an *absolute minimizer for  $F$  on  $U$*  if for every open subset  $V \subset \subset U$  and  $v \in \operatorname{Lip}_{\operatorname{loc}}(V) \cap C(\overline{V})$  with  $u|_{\partial V} = v|_{\partial V}$ , we have  $\mathbb{F}(u, V) \leq \mathbb{F}(v, V)$ , that is,

$$\operatorname{ess\,sup}_{x \in V} F(x, \nabla u(x)) \leq \operatorname{ess\,sup}_{x \in V} F(x, \nabla v(x)).$$

Moreover, given a function  $f \in \operatorname{Lip}(\partial U)$ ,  $u \in \operatorname{Lip}_{\operatorname{loc}}(U) \cap C(\overline{U})$  is called an *absolutely minimizing Lipschitz extension* of  $f$  on  $U$  with respect to  $F$  if  $u$  is an absolute minimizer for  $F$  on  $U$  and  $u|_{\partial U} = f$ . In literature, an absolute minimizer of the model case (1.2) is also termed as an *infinity harmonic function* in  $U$ .

Aronsson [4] proved the existence of absolute minimizers for (1.2) with given Lipschitz Dirichlet boundary data. His approach is as follows: for a given bounded domain  $\Omega \subset \mathbb{R}^n$  and  $g \in \operatorname{Lip}(\partial \Omega)$ , find the "best" Lipschitz extension of  $g$  to  $\Omega$ . By the best extension, we mean that the function should satisfy the condition

$$L_u(V) = L_u(\partial V) \quad \text{for all } V \subset \Omega, \quad (1.3)$$

where  $L_u(E) := \sup_{x, y \in E} \frac{u(y) - u(x)}{|y - x|}$  denotes the smallest Lipschitz constant of  $u$  in a set  $E$ . A brief review of the motivation of this approach can be found in Jensen's seminal work [25]. Notice that for any function  $v$  in any domain  $V$  we have  $L_v(V) \geq L_v(\partial V)$ . It is not a trivial work to attain the equality in (1.3). We can easily find the following Lipschitz

extensions of  $g$  by the McShane(-Whitney) extension

$$\begin{aligned}\Psi(x) &:= \inf_{y \in \partial\Omega} \left( g(y) + L_g(\partial\Omega)|y - x| \right) \\ \Phi(x) &:= \inf_{y \in \partial\Omega} \left( g(y) - L_g(\partial\Omega)|y - x| \right).\end{aligned}$$

But  $\Psi$  and  $\Phi$  do not satisfy (1.3) except  $\Psi \equiv \Phi$ ; see [4]. It turns out that the functions which satisfy (1.3) are exactly the ones that are infinity harmonic; see [6].

Aronsson [4] also formally derived the infinity Laplace equation

$$\Delta_\infty u(x) := (D^2u(x)\nabla u(x)) \cdot \nabla u(x) = 0, \quad (1.4)$$

as the Euler-Lagrange equation of the variational problem (1.2), where  $D^2u(x)$  denotes the Hessian of  $u$  at  $x$ . Aronsson proved that a  $C^2$ -solution  $u$  is infinity harmonic if and only if it satisfies (1.4). Of course at that time, he did not have the right tools to interpret the equation (1.4) for non-smooth functions. This was a major problem since there are non-smooth infinity harmonic functions, such as  $u(x, y) = y^{4/3} - x^{4/3}$  in the plane.

After the development of the viscosity solution theory by Crandall and Lions in the 1980's, Jensen [25] proved that infinity harmonic functions are viscosity solutions to equation (1.4) and vice versa, under given Dirichlet boundary data. He also proved the existence and the uniqueness results of infinity harmonic functions under more general Dirichlet boundary data. As already remarked by Jensen [25], his uniqueness approach uses equation (1.4) intensively. Thus, it seems hard to extend his uniqueness approach to more general cases in which one can not derive an equation of type (1.4) from the  $L^\infty$  variational problem.

More recent proofs for the uniqueness of Jensen [25] can be found in Crandall, Gunnarsson and Wang [11], Barles and Busca [7], and Armstrong and Smart [1]. Among these proofs, the key idea, to derive the uniqueness result for (1.2), is to use the characterization of infinity harmonic functions via comparison with cones, which was first properly stated by Crandall, Evans and Gariepy [10]. To gain some intuition, observe that for all  $a > 0$ , the cone function  $C(x) := a|x|$  is a smooth solution of the infinity Laplace equation (1.4) in  $\mathbb{R}^n \setminus \{0\}$ . This can be easily seen by noticing that

$$|\nabla C(x)| = a \quad \text{for all } x \neq 0. \quad (1.5)$$

Differentiating (1.5), one easily obtains  $\Delta_\infty C(x) = 0$  in  $\mathbb{R}^n \setminus \{0\}$ . In this regard, the cone function is a sort of fundamental solutions to the infinity Laplace equation and the comparison with cones is a sort of (weak) comparison principles. In [10], a very elegant proof is used to show the equivalence of being infinity harmonic and satisfying the comparison with cones. We would like to point out that the comparison with cones has turned out to be a fruitful point of view, and for example, Savin's proof [32] for  $C^1$  regularity of infinity harmonic functions in the plane is entirely based on cones; see also [17, 34].

Our main aim of this paper is to consider the existence and the uniqueness results for the minimization problem (1.1) associated with a very general Finsler structure  $F$ . The typical feature is that we impose very less regularity on  $F$ . In particular, in our case, there is no PDE associated to the variational problem (1.1) and hence standard techniques from elliptic PDEs are not available. Our main result reads as follows.

**Theorem 1.1.** *Let  $F : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$  be an admissible Finsler structure on  $\Omega$ . Then for each open subset  $U \subset\subset \Omega$  and each boundary data  $f \in \text{Lip}(\partial U)$ , there exists a unique absolutely minimizing Lipschitz extension on  $U$  with respect to  $F$ .*

When  $F(x, v) := \langle A(x)v, v \rangle$ , where  $A$  is a diffusion matrix-valued function, Theorem 1.1 reduces to Theorem 5 in [30]. Thus, our main result can be regarded as a natural generalization of [30, Theorem 5] from the diffusion case to more general Finsler case.

Although the general principle behind the proof of Theorem 1.1 is similar to [30, Theorem 5], our approach (for the existence) is substantially different from [30]. Indeed, the proof given in [30] depends heavily on the speciality of the structure  $F(x, v) := \langle A(x)v, v \rangle$  and seems not to be easily generalisable to our case.

For the existence result, our proof relies on the (crucial) Lemma 4.2 and Proposition 3.1, which allows us to describe the absolute minimizer via the pointwise Lipschitz constant. In this step, we also borrow some ideas from the recent related work [23], which allow us to relate the geometric and the analytic aspects of admissible Finsler structures.

For the uniqueness result, we follow closely the idea of [1] and [30], that is, we first characterize absolute minimizers for the variational problem (1.1) via comparison with cones, very similar to the infinity Laplace case. Then we establish the comparison with cones as in [1]. Somewhat surprisingly, we do not really need equations (as in the infinity Laplace case) to effectively use the comparison with cones.

We remark here that there are many natural questions that can be done after this work. First of all, one could consider the linear approximation property for admissible Finsler structures with extra smoothness assumption as in [30, Section 6] and the regularity issues as in [18] and [33]. The second possible direction is to generalize these results to certain metric measure spaces as in [27, 28, 29].

This paper is organized as follows. In Section 2, we recall some preliminary results on admissible Finsler structures and the associated (intrinsic) distances. In Section 3, we prove one of the key results, namely, Proposition 3.1. The proof of Theorem 1.1 is contained in Section 4, as a special case of the more general Theorem 4.1. An alternative proof of Lemma 2.5 is provided in the appendix.

Throughout the paper, we use  $|\cdot|$  and  $\langle \cdot, \cdot \rangle$  to denote the standard norm and inner product of Euclidean spaces.

## 2. PRELIMINARIES

**2.1. Finsler structure and its dual.** Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a domain. A Finsler structure on  $\Omega$  is defined as follows.

**Definition 2.1** (Finsler structure). We say that a function  $F : \Omega \times \mathbb{R}^n \rightarrow [0, \infty)$  is a Finsler structure on  $\Omega$  if

- $F(\cdot, v)$  is Borel measurable for all  $v \in \mathbb{R}^n$ ,  $F(x, \cdot)$  is continuous for a.e.  $x \in \Omega$ ;
- $F(x, v) > 0$  for a.e.  $x$  if  $v \neq 0$ ;
- $F(x, \lambda v) = |\lambda|F(x, v)$  for all  $\lambda \in \mathbb{R}$  and  $(x, v) \in \Omega \times \mathbb{R}^n$ .

It turns out to be too general for us to study the  $L^\infty$ -variational problem (1.1) in the context of Finsler structures. We will restrict ourselves to the class of admissible Finsler structures.

**Definition 2.2** (Admissible Finsler structure). A Finsler structure  $F$  on  $\Omega$  is said to be admissible if

- $F(x, \cdot)$  is convex for a.e.  $x \in \Omega$ ;
- $F$  is locally equivalent to the Euclidean norm. That is, there exists a continuous function  $\lambda : \Omega \rightarrow [1, \infty)$  such that

$$\lambda(x)^{-1}|v| \leq F(x, v) \leq \lambda(x)|v|.$$

For any admissible Finsler structure  $F$  on  $\Omega$ , we introduce the dual  $F^* : \Omega \times \mathbb{R}^n \rightarrow [0, \infty)$  of  $F$  in the following standard way.

**Definition 2.3** (Dual Finsler structure). Let  $F$  be an admissible Finsler structure on  $\Omega$ . We define  $F^* : \Omega \times \mathbb{R}^n \rightarrow [0, \infty)$ , the dual Finsler structure of  $F$  on  $\Omega$ , by

$$F^*(x, w) := \sup_{v \in \mathbb{R}^n} \left\{ \langle v, w \rangle : F(x, v) \leq 1 \right\}.$$

We remark that it is direct to verify that

$$F^*(x, w) = \max_{v \in \mathbb{R}^n \setminus \{0\}} \left\langle w, \frac{v}{F(x, v)} \right\rangle. \quad (2.1)$$

We shall need the following result on the properties of dual Finsler structures, which can be found in [21, Section 1.2].

**Proposition 2.4** (Basic properties of dual Finsler structures). *Let  $F$  be an admissible Finsler structure on  $\Omega$ . Then the dual Finsler structure  $F^*$  has the following properties:*

- $F^*(\cdot, v)$  is Borel measurable for all  $v \in \mathbb{R}^n$  and  $F^*(x, \cdot)$  is Lipschitz continuous for a.e.  $x \in \Omega$ ;
- $F^*(x, \cdot)$  is a norm for a.e.  $x \in \Omega$ ;
- Let  $\lambda(\cdot)$  be defined as in Definition 2.2. Then  $F^*(x, \cdot)$  satisfies that

$$\lambda(x)^{-1}|v| \leq F^*(x, v) \leq \lambda(x)|v|;$$

- $(F^*)^*(x, v) = F(x, v)$  for all  $(x, v) \in \Omega \times \mathbb{R}^n$ .

**2.2. Intrinsic distance associated to an admissible Finsler structure.** For any admissible Finsler structure  $F$  on  $\Omega$ , we associate  $\Omega$  with an intrinsic distance by setting

$$d_c^F(x, y) := \sup_N \inf_{\gamma \in \Gamma_N^{x,y}} \left\{ \int_0^1 F(\gamma(t), \gamma'(t)) dt \right\} \quad \text{for all } x, y \in \Omega,$$

where the supremum is taken over all subsets  $N$  of  $\Omega$  such that  $|N| = 0$  and  $\Gamma_N^{x,y}(\Omega)$  denotes the set of all Lipschitz continuous curves  $\gamma$  in  $\Omega$  with end points  $x$  and  $y$  such that  $\mathcal{H}^1(N \cap \gamma) = 0$  with  $\mathcal{H}^1$  being the one dimensional Hausdorff measure. Similarly, we define the distance  $d_c^{F^*}$  by

$$d_c^{F^*}(x, y) := \sup_N \inf_{\gamma \in \Gamma_N^{x,y}} \left\{ \int_0^1 F^*(\gamma(t), \gamma'(t)) dt \right\} \quad \text{for all } x, y \in \Omega.$$

For notational simplicity, we write  $d_c^* = d_c^{F^*}$  below. We also need the following intrinsic distance function  $\delta_F$  defined as

$$\delta_F(x, y) := \sup \{u(x) - u(y) : u \in \text{Lip}(\Omega), \|F(x, \nabla u)\|_{L^\infty(\Omega)} \leq 1\}.$$

For the definition of the class of intrinsic distances, we refer to [15, Section 3] or [21, Section 1.1], and it will not play any role in this paper.

The following lemma implies that  $\delta_F$  is actually the same as  $d_c^*$  at infinitesimal scale. The proof can be found in [14, Theorem 3.7]. In the special case when  $F$  is (weak) upper semicontinuous, a simpler proof can be found in [23, Proposition 3.1].

**Lemma 2.5.** *Let  $F$  be an admissible Finsler structure on  $\Omega$ . Then for a.e.  $x \in \Omega$ , we have*

$$\lim_{y \rightarrow x} \frac{\delta_F(x, y)}{d_c^*(x, y)} = 1.$$

**2.3. Comparison of metric derivatives.** For any distance  $d$  on  $\Omega$  and any Lipschitz continuous (with respect to  $d$ ) curve  $\gamma : [a, b] \rightarrow \Omega$ , the length of  $\gamma$  with respect to  $d$  is denoted by  $\mathcal{L}_d(\gamma)$ , that is,

$$\mathcal{L}_d(\gamma) := \sup \left\{ \sum_{i=1}^k d(\gamma(t_i), \gamma(t_{i+1})) \right\},$$

where the supremum is taken over all partitions  $\{[t_i, t_{i+1}]\}$  of  $[a, b]$ .

For any intrinsic distance  $d$ , which is locally equivalent to the Euclidean distance, we define

$$\Delta_d(x, v) := \limsup_{t \rightarrow 0} \frac{d(x, x + tv)}{t} \quad (2.2)$$

for all  $x \in \Omega$  and  $v \in \mathbb{R}^n$ . It turns out that  $\Delta_d$  is a convex Finsler metric. Moreover, it can be proved that for every Lipschitz continuous curve  $\gamma : [a, b] \rightarrow \Omega$ , we have

$$\mathcal{L}_d(\gamma) = \int_a^b \Delta_d(\gamma, \gamma') dt.$$

These facts can be found for instance in [21, Section 1.1].

By [21, Proposition 1.6], for an admissible Finsler structure  $F$ , one always has

$$\Delta_{d_c^*}(x, v) \leq F^*(x, v). \quad (2.3)$$

However, for general Finsler structures, the strict inequality above can hold; see e.g. [15, Example 5.1].

### 3. WEAK COINCIDENCE OF DIFFERENTIAL STRUCTURE AND DISTANCE STRUCTURE

The following proposition is crucial in the proof of the existence part of Theorem 4.1 in Section 4. For its proper formulation, we recall that the pointwise Lipschitz constant function  $\text{Lip}_d u$  of a Borel function  $u : \Omega \rightarrow \mathbb{R}$  with respect to a distance  $d$  is defined as

$$\text{Lip}_d u(x) := \limsup_{y \rightarrow x} \frac{|u(y) - u(x)|}{d(x, y)}.$$

**Proposition 3.1.** *Let  $F$  be an admissible Finsler structure on  $\Omega$ . Then for every open set  $V \subset \subset \Omega$  and every function  $u \in \text{Lip}_{\text{loc}}(\Omega)$ , we have*

$$\text{ess sup}_{x \in V} F(x, \nabla u(x)) = \text{ess sup}_{x \in V} \text{Lip}_{\delta_F} u(x) = \sup_{x \in V} \text{Lip}_{\delta_F} u(x).$$

*Proof.* Let  $V \subset \subset \Omega$  and  $u \in \text{Lip}_{\text{loc}}(\Omega)$  be an arbitrary Lipschitz continuous function. We first show that

$$\text{ess sup}_{x \in V} F(x, \nabla u(x)) \geq \sup_{x \in V} \text{Lip}_{\delta_F} u(x). \quad (3.1)$$

Since both sides of (3.1) are positively 1-homogeneous with respect to  $u$ , we only need to show that if  $\text{ess sup}_{x \in V} F(x, \nabla u(x)) \leq 1$ , then  $\sup_{x \in V} \text{Lip}_{\delta_F} u(x) \leq 1$ .

By the definition of  $\delta_F$ , if  $\text{ess sup}_{x \in V} F(x, \nabla u(x)) \leq 1$ , then  $|u(x) - u(y)| \leq \delta_F(x, y)$  for all  $x, y \in V$ , which implies that

$$\sup_{x \in V} \text{Lip}_{\delta_F} u(x) \leq 1.$$

Thus, we obtain (3.1).

We next show that

$$\text{ess sup}_{x \in V} F(x, \nabla u(x)) \leq \text{ess sup}_{x \in V} \text{Lip}_{\delta_F} u(x). \quad (3.2)$$

Since both sides of (3.2) are positively 1-homogeneous with respect to  $u$ , we only need to show that for a.e.  $x \in V$ , if  $\text{Lip}_{\delta_F} u(x) \leq 1$ , then  $F(x, \nabla u(x)) \leq 1$ .

Note that by Lemma 2.5,  $\text{Lip}_{\delta_F} u(x) = \text{Lip}_{d_c^*} u(x)$  for a.e.  $x \in \Omega$ . Fix such an  $x$ . For each  $v \in \mathbb{R}^n$ , we have

$$\begin{aligned} \langle \nabla u(x), v \rangle &= \lim_{t \rightarrow 0} \frac{u(x + tv) - u(x)}{t} \\ &\leq \limsup_{t \rightarrow 0} \frac{d_c^*(x + tv, x)}{t} \limsup_{t \rightarrow 0} \frac{u(x + tv) - u(x)}{d_c^*(x + tv, x)} \\ &\leq \Delta_{d_c^*}(x, v) \text{Lip}_{d_c^*} u(x) \\ &\leq \Delta_{d_c^*}(x, v) \leq F^*(x, v), \end{aligned}$$

where the last inequality follows from (2.3). Therefore, by Proposition 2.4 we have

$$F(x, \nabla u(x)) = F^{**}(x, \nabla u(x)) = \sup_{v \neq 0} \left\langle \nabla u(x), \frac{v}{F^*(x, v)} \right\rangle \leq 1.$$

This proves (3.2).

Combining (3.1) and (3.2) gives us that

$$\begin{aligned} \text{ess sup}_{x \in V} F(x, \nabla u(x)) &\leq \text{ess sup}_{x \in V} \text{Lip}_{\delta_F} u(x) \\ &\leq \sup_{x \in V} \text{Lip}_{\delta_F} u(x) \\ &\leq \text{ess sup}_{x \in V} F(x, \nabla u(x)), \end{aligned}$$

which implies that all the inequalities above are actually equalities. The proof of Proposition 3.1 is complete.  $\square$

## 4. EXISTENCE AND UNIQUENESS

Let  $F$  be an admissible Finsler structure on  $\Omega$  and  $U \subset\subset \Omega$ . In this section we prove the following theorem.

**Theorem 4.1.** (i) *For every  $f \in \text{Lip}(\partial U)$ , there exists a unique absolutely minimizing Lipschitz extension on  $U$  with respect to  $F$ .*

(ii) *The absolute minimizer is completely determined by the intrinsic distance in the following sense: let  $\delta_F$  and  $\delta_{\tilde{F}}$  be the intrinsic distance associated with the admissible Finsler structures  $F$  and  $\tilde{F}$ , respectively. If for almost all  $x \in U$  there holds*

$$\lim_{x \neq y \rightarrow x} \frac{\delta_F(x, y)}{\delta_{\tilde{F}}(x, y)} = 1, \quad (4.1)$$

*then  $u$  is an absolute minimizer on  $U$  for  $F$  if and only if  $u$  is an absolute minimizer on  $U$  for  $\tilde{F}$ .*

Note that Theorem 1.1 is the first part of Theorem 4.1. The proof of Theorem 4.1 is long and thus is divided into several lemmas. In the following, we first prove the existence part, and then the uniqueness part of Theorem 4.1(i). In the end of this section, we give a complete proof of Theorem 4.1.

**4.1. Proof of existence.** The following lemma is an analogy of [30, Lemma 7], which characterizes absolute minimizers via intrinsic distances.

**Lemma 4.2.** *Let  $u \in \text{Lip}_{\text{loc}}(U)$ . Then  $u$  is an absolute minimizer on  $U$  if and only if for each bounded open subset  $V \subset\subset U$  and all  $v \in \text{Lip}_{\text{loc}}(V) \cap C(\bar{V})$  with  $u|_{\partial V} = v|_{\partial V}$ , one (or both) of the following holds:*

- (i)  $\text{ess sup}_{x \in V} \text{Lip}_{\delta_F} u(x) \leq \text{ess sup}_{x \in V} \text{Lip}_{\delta_F} v(x);$
- (ii)  $\sup_{x \in V} \text{Lip}_{\delta_F} u(x) \leq \sup_{x \in V} \text{Lip}_{\delta_F} v(x).$

*Proof.* In view of Proposition 3.1, Lemma 4.2 is no more than a restatement of the definition of absolute minimizers.  $\square$

Notice that our concept of absolutely minimizing Lipschitz extensions defined in Section 1 corresponds to the strongly absolutely minimizing Lipschitz extension in [27]. Applying Lemma 4.2 and [27, Theorem 3.1], we have the following existence result.

**Lemma 4.3.** *For every  $f \in \text{Lip}(\partial U)$ , there exists an absolutely minimizing Lipschitz extension of  $f$  on  $U$ .*

**4.2. Proof of uniqueness.** We point out here that the existence and the uniqueness of absolutely minimizing Lipschitz extensions in domains in a length space have already been proven in [31] via a probabilistic approach called the Tug-of-War. Here to prove the uniqueness result, we derive the following comparison principle, by applying the strategy developed by Armstrong and Smart [1].

**Lemma 4.4.** *Let  $u, v \in \text{Lip}_{\text{loc}}(U) \cap C(\bar{U})$  be absolute minimizers on  $U$ . Then*

$$\max_{x \in \bar{U}} [u(x) - v(x)] = \max_{x \in \partial U} [u(x) - v(x)].$$



Before going into the proof of Lemma 4.4, let us first recall the definition of the comparison with cones introduced by Crandall et al. [10]. A function  $u \in C(U)$  is said to satisfy the *property of comparison with cones* if for all subsets  $V \subset\subset U$  and all  $a \geq 0$ ,  $b \in \mathbb{R}$  and  $x_0 \in \mathbb{R}^n \setminus V$ , we have

$$(I) \max_{x \in \partial V} [u(x) - C_{b,a,x_0}(x)] \leq 0 \text{ implies } \max_{x \in V} [u(x) - C_{b,a,x_0}(x)] \leq 0;$$

$$(II) \max_{x \in \partial V} [u(x) - C_{b,-a,x_0}(x)] \geq 0 \text{ implies } \max_{x \in V} [u(x) - C_{b,-a,x_0}(x)] \geq 0,$$

where the cone function  $C_{b,a,x_0}$  is defined as

$$C_{b,a,x_0}(x) := b + a \delta_F(x, x_0).$$

It is known that an absolute minimizer satisfies the comparison property with cones; see [10] for Euclidean case and [6, 27, 20, 8, 16] for the setting of metric spaces that are length spaces.

The following is a list of equivalent characterizations for absolute minimizers.

**Lemma 4.5.** *The following statements are mutually equivalent:*

- (i)  $u$  is an absolute minimizer on  $U$ .
- (ii) for all open sets  $V \subset\subset U$ ,  $\text{Lip}_{\delta_F}(u, V) = \text{Lip}_{\delta_F}(u, \partial V)$ .
- (iii)  $u$  satisfies the property of comparison with cones.

*Proof.* (i) $\Rightarrow$ (ii). It is a consequence of Lemma 4.2. Indeed, notice that for every pair  $x, y \in \partial V$  with  $x \neq y$ , by the continuity of  $\delta_F$  we can find  $x_n, y_n \in V$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . By the continuity of  $u$ , we have

$$\frac{|u(x_n) - u(y_n)|}{\delta_F(x_n, y_n)} \rightarrow \frac{|u(x) - u(y)|}{\delta_F(x, y)} \quad \text{as } n \rightarrow \infty.$$

Hence  $\text{Lip}_{\delta_F}(u, V) \geq \text{Lip}_{\delta_F}(u, \partial V)$  holds. Thus, it suffices to prove that  $\text{Lip}_{\delta_F}(u, V) \leq \text{Lip}_{\delta_F}(u, \partial V)$ .

For  $x \in \mathbb{R}^n$ , let

$$w(x) := \sup_{z \in \partial V} [u(z) + \text{Lip}_{\delta_F}(u, \partial V) \delta_F(x, z)].$$

Then  $\text{Lip}_{\delta_F}(w, \mathbb{R}^n) = \text{Lip}_{\delta_F}(u, \partial V)$  and  $w = u$  on  $\partial V$ . Applying Lemma 4.2, we have

$$\sup_{x \in V} \text{Lip}_{\delta_F} u(x) \leq \sup_{x \in V} \text{Lip}_{\delta_F} w(x) \leq \text{Lip}_{\delta_F}(u, \partial V).$$

Notice that  $(U, \delta_F)$  is a geodesic space. Indeed, since  $U \subset\subset \Omega$  is open bounded, our assumption on  $F$  implies that there exist positive constants  $\alpha := \alpha(U)$  and  $\beta := \beta(U)$  such that

$$\alpha|v| \leq F(x, v) \leq \beta|v|$$

for all  $x \in U$  and  $v \in \mathbb{R}^n$ . Combining this fact together with [21, Theorem 3.9] yields that  $(U, \delta_F)$  is a geodesic space. Thus, given a pair of points  $x, y \in U$ , we can select a  $\delta_F$ -geodesic curve  $\gamma$  joining  $x$  and  $y$ .

If  $\gamma \subset V$ , then

$$\begin{aligned} |u(x) - u(y)| &\leq \int_{\gamma} \text{Lip}_{\delta_F} u(z) \, ds \\ &\leq \delta_F(x, y) \sup_{x \in V} \text{Lip}_{\delta_F} u(x) \\ &\leq \delta_F(x, y) \text{Lip}_{\delta_F}(u, \partial V). \end{aligned}$$

Here  $ds$  denotes arc-length integral on  $\gamma$  with respect to the metric  $\delta_F$ . If  $\gamma \not\subset V$ , denote by  $\hat{x}, \hat{y} \in \gamma \cap \partial V$  points that have shortest distance to  $x$  and  $y$ , respectively. Then

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(\hat{x})| + |u(\hat{x}) - u(\hat{y})| + |u(\hat{y}) - u(y)| \\ &\leq [\delta_F(x, \hat{x}) + \delta_F(y, \hat{y})] \sup_{x \in V} \text{Lip}_{\delta_F} u(x) + \delta_F(\hat{x}, \hat{y}) \text{Lip}_{\delta_F}(u, \partial V) \\ &\leq \delta_F(x, y) \text{Lip}_{\delta_F}(u, \partial V). \end{aligned}$$

Thus, in both cases, we have the estimate

$$\frac{|u(x) - u(y)|}{\delta_F(x, y)} \leq \text{Lip}_{\delta_F}(u, \partial V),$$

which implies that  $\text{Lip}_{\delta_F}(u, V) \leq \text{Lip}_{\delta_F}(u, \partial V)$ .

(ii) $\Rightarrow$ (iii). We prove (I) by a contradiction argument. The proof of (II) is similar (and left to the interested reader). Let  $u$  be an absolute minimizer and assume that

$$\max_{x \in \partial V} [u(x) - C_{b,a,x_0}(x)] \leq 0.$$

Suppose that (I) fails, that is,  $\max_{x \in V} [u(x) - C_{b,a,x_0}(x)] > 0$ . Denote by  $W$  the open set of all  $x \in V$  such that  $u(x) > C_{b,a,x_0}(x)$ . By assumption,  $W$  is not empty. Moreover, we have  $u = C_{b,a,x_0}$  on  $\partial W$ . Since  $W \subset V \subset U$ , by assumption (ii) we have

$$\text{Lip}_{\delta_F}(u, \overline{W}) = \text{Lip}_{\delta_F}(u, W) = \text{Lip}_{\delta_F}(u, \partial W) = a.$$

For  $x \in W$ , let  $\gamma$  be a  $\delta_F$ -geodesic curve joining  $x$  and  $x_0$ , and take  $z \in \partial W \cap \gamma$  be a closest point to  $x$ . Then

$$\begin{aligned} u(x) - u(z) &> C_{b,a,x_0}(x) - C_{b,a,x_0}(z) \\ &= a\delta_F(x_0, x) - a\delta_F(x_0, z) \\ &= a\delta_F(x, z), \end{aligned}$$

which implies that  $\text{Lip}_{\delta_F}(u, W) > a$ . We reach a contradiction. So  $W$  must be empty. Therefore (ii) implies (iii).

(iii) $\Rightarrow$ (i). We only need to notice that, with the help of Proposition 3.1, the argument provided by the proof of [27, Proposition 5.8] still works here, without the additional weak Fubini property required in [27]; see also [6]. This completes the proof of Lemma 4.5.  $\square$

With the aid of Lemmas 4.2 and 4.5, Lemma 4.4 will be proved by following the procedure from [1]. Since the proof in [1] is for the case  $F(x, \cdot) := |\cdot|$ , we write down the details below for the reader's convenience. We need some notation. For all  $r > 0$ , let

$$U_r := \{z \in U : \overline{B_{\delta_F}(z, r)} \subset U\}.$$

For  $x \in U_r$ , we let

$$S_r^+ u(x) := \frac{u^r(x) - u(x)}{r} \quad \text{and} \quad S_r^- u(x) := \frac{u(x) - u_r(x)}{r},$$

where  $u^r(x) := \sup_{\delta_F(z, x) \leq r} u(z)$  and  $u_r(x) := \inf_{\delta_F(z, x) \leq r} u(z)$ .

*Proof of Lemma 4.4.* First we claim that for  $x \in U_{2r}$ , we have

$$S_r^- u^r(x) - S_r^+ u^r(x) \leq 0 \leq S_r^- v_r(x) - S_r^+ v_r(x). \quad (4.2)$$

Indeed, let  $y \in \overline{B_{\delta_F}(x, r)}$  and  $z \in \overline{B_{\delta_F}(x, 2r)}$  such that  $u^r(x) = u(y)$  and  $(u^r)^r(x) = u^{2r}(x) = u(z)$ . Observe that  $(u^r)_r(x) \geq u(x)$ . Then we have

$$\begin{aligned} S_r^- u^r(x) - S_r^+ u^r(x) &= \frac{1}{r} [2u^r(x) - (u^r)^r(x) - (u^r)_r(x)] \\ &\leq \frac{1}{r} [2u(y) - u(z) - u(x)]. \end{aligned} \quad (4.3)$$

Note that for  $w \in \Omega$  such that  $\delta_F(x, w) = 2r$ , we have

$$u(w) \leq u(z) = u(x) + [u(z) - u(x)] = u(x) + \frac{[u(z) - u(x)]}{2r} \delta_F(w, x).$$

Thus, the comparison with cones property of  $u$  implies that the inequality

$$u(w) \leq u(x) + \frac{[u(z) - u(x)]}{2r} \delta_F(w, x)$$

holds for all  $w \in \Omega$  with  $\delta_F(x, w) \leq 2r$ . In particular, taking  $w = y$  and noticing that  $\delta_F(y, x) \leq r$ , we obtain

$$\begin{aligned} u(y) &\leq u(x) + \frac{[u(z) - u(x)]}{2r} \delta_F(y, x) \\ &\leq u(x) + \frac{1}{2} [u(z) - u(x)] \\ &= \frac{1}{2} [u(z) + u(x)], \end{aligned}$$

which, together with (4.3), implies the first inequality of (4.2). The second inequality of (4.2) follows similarly.

Next we claim that (4.2) gives us that

$$\sup_{x \in U_r} [u^r(x) - v_r(x)] = \sup_{x \in U_r \setminus U_{2r}} [u^r(x) - v_r(x)] \quad (4.4)$$

for all  $r > 0$ . Once we prove that (4.4) holds, then Lemma 4.4 follows by letting  $r \rightarrow 0$  in (4.4). Thus, we only need to prove (4.4).

Suppose, on the contrary, that (4.4) dose not hold. Then there exists some  $r > 0$  for which

$$\sup_{x \in U_r} [u^r(x) - v_r(x)] > \sup_{x \in U_r \setminus U_{2r}} [u^r(x) - v_r(x)]. \quad (4.5)$$

By the continuity of  $u^r - v_r$ , there must exist some  $y \in \overline{U_r}$  such that

$$u^r(y) - v_r(y) = \sup_{x \in U_r} [u^r(x) - v_r(x)].$$

Note that (4.5) implies that  $y \in U_{2r}$ . Denote by  $E$  the set of all such  $y$  and let

$$K := \left\{ x \in E : u^r(x) = \max_{z \in E} u^r(z) \right\}.$$

Then  $K$  is a closed subset of  $U_{2r}$  by the continuity of  $u^r$  again. Choose  $x_0 \in \partial K$ . Since  $x_0 \in E$ , for every  $x \in U_r$  we have

$$u^r(x_0) - v_r(x_0) \geq u^r(x) - v_r(x).$$

Since  $x_0 \in U_{2r}$ , we have  $B_{\delta_F}(x_0, r) \subset U_r$ . Thus, for every  $x \in B_{\delta_F}(x_0, r)$ , we deduce from above inequality that

$$u^r(x_0) - v_r(x_0) \geq \inf_{z \in B_{\delta_F}(x_0, r)} u^r(z) - v_r(x) = (u^r)_r(x_0) - v_r(x).$$

That is,

$$u^r(x_0) - (u^r)_r(x_0) \geq v_r(x_0) - v_r(x).$$

Divide by  $r$  on each side of above equality and then take the infimum over  $x \in B_{\delta_F}(x_0, r)$ . We obtain

$$S_r^- u^r(x_0) \geq S_r^- v_r(x_0). \quad (4.6)$$

Now we have two cases.

*Case 1:*  $S_r^+ u^r(x_0) = 0$ .

*Case 2:*  $S_r^+ u^r(x_0) > 0$ .

Consider *Case 1*. In this case, (4.2) yields that  $S_r^- u^r(x_0) \leq 0$ . Hence we derive that  $S_r^- u^r(x_0) = 0$  holds, which, together with (4.6), implies that  $S_r^- v_r(x_0) = 0$ . By (4.2) again, we have  $S_r^+ v_r(x_0) \leq 0$  and hence  $S_r^+ v_r(x_0) = 0$ . So we obtain  $u^r \equiv u^r(x_0)$  and  $v_r \equiv v_r(x_0)$  hold on  $B_{\delta_F}(x_0, r)$ . This contradicts to the fact that  $x_0 \in \partial K$ .

It remains to consider *Case 2*. Choose  $z \in \overline{B_{\delta_F}(x_0, r)}$  such that

$$0 < r S_r^+ u^r(x_0) = u^r(z) - u^r(x_0).$$

Since  $u^r(z) > u^r(x_0)$  and  $x_0 \in K$ , it follows that  $z \notin E$ . Note that  $z \in U_r$  since  $x_0 \in U_{2r}$ . Therefore the fact that  $x_0 \in E$  yields

$$u^r(x_0) - v_r(x_0) > u^r(z) - v_r(z).$$

That is, we have

$$v_r(z) - v_r(x_0) > u^r(z) - u^r(x_0).$$

Hence we derive that

$$r S_r^+ v_r(x_0) \geq v_r(z) - v_r(x_0) > u^r(z) - u^r(x_0) = r S_r^+ u^r(x_0),$$

which, together with (4.6), implies that

$$S_r^+ v_r(x_0) - S_r^- v_r(x_0) > S_r^+ u^r(x_0) - S_r^- u^r(x_0).$$

We obtain a contradiction to (4.2).

Since both cases above do not hold, we conclude that (4.5) is not true for any  $r > 0$ . That is, (4.4) holds. The proof of Lemma 4.4 is complete.  $\square$

Now we are in a position to prove Theorem 4.1.

*Proof of Theorem 4.1.* Theorem 4.1(i) follows from Lemmas 4.3 and 4.4. Theorem 4.1(ii) follows from Lemma 4.2 with the observation that under the assumption (4.1),

$$\text{Lip}_{\delta_F} u = \text{Lip}_{\delta_{\bar{F}}} u$$

almost everywhere for every  $u \in \text{Lip}_{\text{loc}}(\mathbb{R}^n)$ . The proof of Theorem 4.1 is complete.  $\square$

#### APPENDIX: AN ALTERNATIVE PROOF OF LEMMA 2.5

In the appendix, we give an alternative proof of Lemma 2.5 based on an approximation argument similar to the proof of [23, Proposition 3.1]. The proof is based on a personal communication with Professor Andrea Davini. In particular, he draws our attention on the useful reference [13] and carefully explains its relation with [12]. We would like to express our gratitude here for his kind help.

*Proof of Lemma 2.5.* The proof is similar to that of [23, Proposition 3.1]. The inequality  $\delta_F(x, y) \leq d_c^*(x, y)$  follows directly from definition. Indeed, for each Lipschitz function  $u$  with  $\|F(x, \nabla u(x))\|_\infty \leq 1$ , each  $x, y \in \Omega$ , for each Lipschitz curve  $\gamma$  joining  $x$  and  $y$  that is transversal to the zero measure set  $N := \{x \in \Omega : F(x, \nabla u(x)) > 1\}$ ,

$$\begin{aligned} u(x) - u(y) &= \int_\gamma \langle \nabla u(\gamma(t)), \gamma'(t) \rangle dt \\ &\leq \int_\gamma F^*(\gamma(t), \gamma'(t)) dt = \mathcal{L}_{d_c^*}(\gamma), \end{aligned}$$

where  $\mathcal{L}_{d_c^*}$  denotes the length of the curve  $\gamma$  with respect to the metric  $d_c^*$ . Taking infimum over all admissible curves on the right-hand side and then supremum over all admissible functions over the left-hand side, we obtain

$$\delta_F(x, y) \leq d_c^*(x, y).$$

In particular,

$$\limsup_{y \rightarrow x} \frac{\delta_F(x, y)}{d_c^*(x, y)} \leq 1.$$

So we are left to prove that

$$\liminf_{y \rightarrow x} \frac{\delta_F(x, y)}{d_c^*(x, y)} \geq 1. \quad (4.7)$$

When  $F$  is continuous, (4.7) holds by [23, Proposition 3.1]. In the general case when  $F$  is only Borel measurable, we can use an approximation argument as follows. Since (4.7) is at infinitesimal scale, we can assume that  $F$  is uniform elliptic with absolute positive constants  $\alpha$  and  $\beta$ . That is,  $\alpha|v| \leq F(x, v) \leq \beta|v|$  for all  $x \in \Omega$  and  $v \in \mathbb{R}^n$ . Then, by [12, Theorem 4.1] or [13, Section 2], there exists a sequence  $\{F_n\}_{n \in \mathbb{N}}$  of continuous Finsler structures such that

$$d_c^{*n} \rightarrow d_c^* \quad \text{and} \quad \limsup_{n \rightarrow \infty} \delta_{F_n} \leq \delta_F$$

with respect to the uniform convergence of distances on  $\Omega \times \Omega$ , where  $\delta_{F_n}$  is the distance induced by  $F_n$  in the same way as that of  $\delta_F$ , and  $d_c^{*n} := d_c^{F_n^*}$  is the distance induced by the dual of the Finsler structure  $F_n$  for all  $n$ .

Now, for any  $\varepsilon > 0$ , there exists a number  $N_0 > 1$  such that for  $n \geq N_0$ , we have

$$\frac{\delta_F(x, y)}{d_c^*(x, y)} \geq (1 - \varepsilon) \frac{\delta_{F_n}(x, y)}{d_c^{*n}(x, y)}.$$

On the other hand, since  $F_n$  is continuous, by [23, Proposition 3.1], we have

$$\liminf_{y \rightarrow x} \frac{\delta_{F_n}(x, y)}{d_c^{*n}(x, y)} \geq 1.$$

Thus, we deduce that

$$\liminf_{y \rightarrow x} \frac{\delta_F(x, y)}{d_c^*(x, y)} \geq \liminf_{y \rightarrow x} (1 - \varepsilon) \frac{\delta_{F_n}(x, y)}{d_c^{*n}(x, y)} \geq 1 - \varepsilon.$$

Sending  $\varepsilon \rightarrow 0$  yields (4.7). The proof of Lemma 2.5 is complete.  $\square$

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